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FEEDBACK BOUNDARY CONTROL PROBLEMS FOR LINEAR SEMIGROUPS

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ABSTRACT

We investigate the abstract Cauchy problem

$$\frac{d}{dt}x(t)=A(I+B)x(t)$$

and apply the obtained generation results to feedback boundary control problems.

1. Introduction and notation

Let Ω be an open subset of $\mathbf{R}^n$ with boundary $\Gamma=\partial\Omega$. Let $(X,\|\cdot\|)$ be a Banach space of functions $\Omega\to\mathbf{R}^n$. A typical boundary control system can be described as

$$\begin{aligned}&\frac{\partial x}{\partial t}(y,t)=\mathscr{A}x(y,t)+Gu(y,t), \qquad t>0,\quad y\in\Omega,\\ &x(y,0)=x_0(y), \qquad y\in\Omega, \qquad\qquad (1.1)\\ &\tau x(y,t)=Fu(y,t), \qquad t>0,\quad y\in\Gamma.\end{aligned}$$

Here $\mathscr{A}$ stands for a linear partial differential operator acting in X, G is a continuous linear operator from a Banach space U of control functions into X and F is a continuous linear operator from U into the trace space of X and τ denotes the linear boundary operator that maps functions defined on Ω onto functions defined on Γ. The control function u is assumed to belong to $L^1_{\mathrm{loc}}(0,\infty;U)$ (or $L^2_{\mathrm{loc}}(0,\infty;U)$).

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In practice, we can treat (1.1) via the semigroup approach as follows.

Let A be a restriction of $\mathscr{A}$ with homogeneous boundary conditions, i.e. for $x \in D(A)$ we have $\tau x = 0$. Then we are naturally led to define a linear operator D that maps the trace of a function into a function defined on Ω by

$$Dv = g \qquad \text{iff } \mathscr{A}v = 0 \quad \text{and} \quad \tau v = g.$$

If this operator is well-defined then we are led to the Cauchy problem

$$\tag{1.2} \begin{aligned} &\frac{d}{dt} x(t) = A(x(t) - DFu(t)), \qquad t > 0, \\ &x(0) = x_0. \end{aligned}$$

Equation (1.2), or, more generally the abstract Cauchy problem

$$\tag{1.3} \frac{d}{dt} x(t) = A(x(t) - \tilde{F}u(t)) + \tilde{G}u(t), \qquad t > 0,$$

is called an *abstract boundary control problem.*

The objective of this paper is devoted to the study of this problem and in particular the important case where the control u is built up in terms of the state $x(\cdot)$, i.e. $u(t) = Kx(t)$. Usually, we have only a finite number of controls available and hence (1.3) becomes

$$\tag{1.3'} \begin{aligned} &\frac{dx}{dt}(t) = A(I + B)x(t) + G_1 x(t), \qquad t > 0, \\ &x(0) = x_0, \end{aligned}$$

where B is a linear (bounded) operator which has finite dimensional range. As G_1 is a continuous linear operator the fundamental problem is to derive conditions such that $A(I + B)$ generates a C_0-semigroup on X. This question is investigated in detail in Section 3 whereas in Section 4 we present some interesting applications.

Applications to optimal control problems as well as related approximation results will be considered in detail in forthcoming papers.

2. The generation theorems

The objective of this section is to derive some conditions that guarantee that the boundary control problem (1.3) is well-posed, i.e. $A(I + B)$ is the infinitesimal generator of a C_0-semigroup on X. To begin with, we recall some well-known basic facts on linear C_0-semigroups:

A family $T(t)$, $t \geqq 0$, of continuous linear operators on X is called a C_0-semigroup if

$T(0) = I$, the identity map on X,

$T(t+s) = T(t)T(s)$ for all nonnegative s, t,

$\lim_{t\to 0^+} T(t)x = x$ for all $x \in X$.

The infinitesimal generator A of $T(\cdot)$ is given by

$$D(A) = \left\{ x \in X \,\middle|\, \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) \text{ exists} \right\},$$

$$Ax = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x).$$

For elementary properties of semigroups and their infinitesimal generators we refer to [2], [3], [7], [9], [16]. In particular, we mention that A is a closed linear operator and hence $D(A)$ becomes a Banach space under the norm $|x|_A = \|x\| + \|Ax\|$. $(D(A), |\cdot|_A)$ will be denoted by X_A.

Moreover, we shall frequently make use of Ball's Theorem ([1]), stating that a closed linear operator C in X is the infinitesimal generator of a C_0-semigroup $T(\cdot)$ iff for each $x \in X$ there exists a unique weak solution $u(t)$ satisfying

$$\langle u(t), x^* \rangle = \langle x, x^* \rangle + \int_0^t \langle u(s), C^* x^* \rangle ds \quad \text{for all } t \geqq 0 \text{ and } x^* \in D(C^*).$$

In this case, we have $u(t) = T(t)x$ for all $t \geqq 0$.

Our main result on generation of semigroups is

THEOREM 2.1. *Let A be the infinitesimal generator of a C_0-semigroup $S(\cdot)$ in X. Let $(Z, |\cdot|_Z)$ be a Banach space so that*

(Za) *Z is continuously embedded in X,*

(Zb) *for all continuous functions $\phi : [0, t] \to Z$*

$$\int_0^t S(t-s)\phi(s)ds \in D(A),$$

(Zc)
$$\left\| A \int_0^t S(t-s)\phi(s)ds \right\| \leqq \gamma(t) \sup_{0 \leqq s \leqq t} |\phi(s)|_Z$$

where $\gamma : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function with $\gamma(0) = 0$.

Let B be a continuous linear operator $X \to Z$. Then $A(I + B)$ is the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X.

For simplicity, we shall say that a Banach space $(Z, |\cdot|_Z)$ satisfies condition (Z) with respect to A iff $(Z\text{a})$–$(Z\text{c})$ hold.

PROOF. We construct the solutions of

$$\frac{d}{dt}x(t) = A(I+B)x(t), \qquad t>0,$$

$$x(0) = x_0,$$

by constructing a fixed-point of

$$x(t) = S(t)x_0 + A\int_0^t S(t-s)Bx(s)ds.$$

Fix $x_0 \in X$ and $\hat{t} > 0$. Then it is easily checked that the map

$$(Tz)(t) = S(t)x_0 + A\int_0^t S(t-s)Bz(s)ds, \qquad 0 \leqq t < \hat{t}$$

maps $C(0,\hat{t}; X)$ into itself, and for all $0 \leqq t < \hat{t}$ we have

$$\begin{aligned}\|(Tz_1)(t) - (Tz_2)(t)\| &\leqq \left| \int_0^t S(t-s)B(z_1(s) - z_2(s))ds \right|_A \\ &\leqq \gamma(t)b \sup_{0\leqq s\leqq t} |z_1(s) - z_2(s)|_Z\end{aligned}$$

where b is the norm of B regarded as an operator $X \to Z$.

Choosing $\hat{t}$ sufficiently small we conclude that there exists a unique fixed point of T. As $b\gamma(t)$ does not depend on x_0, we may continue this procedure and obtain a continuous function $x : [0,\infty) \to X$ satisfying

$$x(t) = S(t)x_0 + A\int_0^t S(t-s)Bx(s)ds.$$

We next set $y(t) = \int_0^t x(s)ds$. Then

$$\begin{aligned} y(t) &= \int_0^t S(s)x_0 ds + \int_0^t A\int_0^s S(s-\tau)Bx(\tau)d\tau ds \\ &= \int_0^t S(s)x_0 ds + A\int_0^t \int_\tau^t S(s-\tau)Bx(\tau)dsd\tau \\ &= \int_0^t S(s)x_0 ds + \int_0^t [S(t-\tau) - I]Bx(\tau)d\tau \\ &= \int_0^t S(s)x_0 ds + \int_0^t S(t-\tau)Bx(\tau)d\tau - B\int_0^t x(\tau)d\tau. \end{aligned}$$

Consequently,

$$(I+B)y(t)=\int_0^t S(\tau)x_0 d\tau+\int_0^t S(t-\tau)Bx(\tau)d\tau$$

and

$$A(I+B)y(t)=x(t)-x_0.$$

Therefore, we obtain for all $z^*\in D((A(I+B))^*)$ (note that the operator $A(I+B)$ is closed)

$$\langle x(t)-x_0, z^*\rangle=\left\langle\int_0^t x(s)ds, (A(I+B))^*z^*\right\rangle$$

which implies that $x(\cdot)$ is a weak solution of

$$\frac{d}{dt}x(t)=A(I+B)x(t).$$

We next have to verify that this weak solution is unique. Putting $y(t)=\int_0^t x(s)ds$ and $x(0)=0$ we obtain for $z^*\in D((A(I+B))^*)$

$$\langle x(t), z^*\rangle=\langle y(t), (A(I+B))^*z^*\rangle$$

which implies that $y(t)\in D(A(I+B))$ and

$$y'(t)=x(t)=A(I+B)y(t).$$

Setting $u(t)=(I+B)y(t)$, finally, gives

$$u'(t)=Au(t)+Bx(t),$$

i.e.

$$u(t)=\int_0^t S(t-s)Bx(s)ds.$$

As $u'(t)=(I+B)x(t)=A\int_0^t S(t-s)Bx(s)ds+Bx(t)$, x is the unique solution of

$$x(t)=A\int_0^t S(t-s)Bx(s)ds$$

which is zero.

Hence for all $x_0\in X$ there exists a unique weak solution and Ball's Theorem ([1]) yields the claim. □

From the practical point of view it is sometimes easier to verify the conditions given in the following theorem:

THEOREM 2.2. *Let $(Z, |\cdot|_Z)$ be a Banach space and A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X so that Z is continuously embedded into X. Then the following assertions are equivalent*:

(i) *Z satisfies assumption (Z) with respect to A,*

(ii) *there exists a continuous, nondecreasing function $\beta:[0,\infty)\to[0,\infty)$ with $\beta(0)=0$ such that for all $y^*\in X^*$ the total variation of $T^*(\cdot)y^*$ on $[0,t]$ taken with respect to the Z^*-norm is bounded by $\beta(t)\|y^*\|$,*

(iii) *there exists a continuous, nondecreasing function $\beta:[0,\infty)\to[0,\infty)$ with $\beta(0)=0$ such that for all $y^*\in D(A^*)$ and $t>0$*

$$\int_0^t \|T^*(s)A^*y^*\|_{Z^*}ds \leqq \beta(t)\|y^*\|.$$

PROOF. To begin with, we show that (i) implies (ii). Let $y^*\in X^*$ and let $0=t_0<t_1<\cdots<t_n=t$ be a partition of $[0,t]$. Fix some $\eta>0$ and choose elements $z_i\in Z$ so that $|z_i|_Z=1$ and

$$|\langle z_i, T^*(t_{i+1})y^*\rangle-\langle z_i, T^*(t_i)y^*\rangle|\geqq\frac{1}{1+\eta}|T^*(t_{i+1})y^*-T^*(t_i)y^*|_{Z^*}.$$

We put $\phi(s)=z_i$ for $t-t_{i+1}\leqq s<t-t_i$. For sufficiently small $\varepsilon>0$ we set

$$\phi_\varepsilon(s)=\begin{cases}\frac{1}{\varepsilon}(s-t+t_{i+1})z_i & t-t_{i+1}\leqq s<t-t_{i+1}+\varepsilon,\\ z_i & t-t_{i+1}+\varepsilon\leqq s<t-t_i<\varepsilon,\\ \frac{1}{\varepsilon}(t-t_i-s)z_i & t-t_i-\varepsilon\leqq s<t-t_i.\end{cases}$$

For all $z\in Z$ we have

$$A\left(\frac{1}{\varepsilon}\int_0^\varepsilon sT(s)zds\right)=A\frac{1}{\varepsilon}\int_0^\varepsilon\int_s^\varepsilon T(\tau)zd\tau ds=\frac{1}{\varepsilon}\int_0^\varepsilon(T(\varepsilon)z-T(\tau)z)d\tau$$

and hence

$$\lim_{\varepsilon\to0^+}A\frac{1}{\varepsilon}\int_0^\varepsilon sT(s)zds=0.$$

Consequently, $A\int_0^t T(t-s)\phi_\varepsilon(s)ds$ converges to $A\int_0^t T(t-s)\phi(s)ds$ as $\varepsilon\to0^+$. Making use of (i) we obtain

$$\left\|A\int_0^t T(t-s)\phi(s)ds\right\|=\lim_{\varepsilon\to0^+}\left\|A\int_0^t T(t-s)\phi_\varepsilon(s)ds\right\|\leqq\gamma(t).$$

On the other hand we have

$$\begin{aligned}\sum_{i=0}^{n-1} | T^*(t_{i+1})y^* - T^*(t_i)y^* |_{Z^*} &\leqq (1+\eta)\sum_{i=0}^{n-1} \langle T(t_{i+1})z_i, y^*\rangle - \langle T(t_i)z_i, y^*\rangle \\ &= (1+\eta)\sum_{i=0}^{n-1}\left\langle A\int_{t_i}^{t_{i+1}} T(s)\phi(t-s)ds, y^*\right\rangle \\ &= (1+\eta)\left\langle A\int_0^t T(s)\phi(t-s)ds, y^*\right\rangle \\ &\leqq (1+\eta)\gamma(t)\| y^*\|.\end{aligned}$$

As η and the sequence (t_i) were arbitrary, we obtain

$$\mathrm{Var}_{Z^*}(T^*(s)y^*; 0 \leqq s \leqq t) \leqq \gamma(t)\| y^*\|.$$

In order to prove that (ii) implies (iii) choose $y^* \in D(A^*)$. Then we have for all $t > 0$

$$\left(\mathrm{w}^*\text{-}\frac{d}{dt}\right) T^*(t)y^* = T^*(t)A^*y^*$$

where $(\mathrm{w}^*\text{-}d/dt)$ denotes the derivation in the weak-star topology. Hence ([3], Appendix)

$$\mathrm{Var}_{Z^*}(T^*(s)y^*; 0 \leqq s \leqq T) = \int_0^t \| T^*(s)A^*y^*\|_{Z^*}ds.$$

Finally, suppose that (iii) holds. Assume that ϕ is continously differentiable $[0,\infty) \to Z$. Then

$$\int_0^t T(t-s)\phi(s)ds \in D(A) \qquad \text{for all } t \geqq 0.$$

Let $y^* \in Y^*$ such that $\| y^*\| = 1$ and

$$\left\| A\int_0^t T(t-s)\phi(s)ds\right\| = \left\langle A\int_0^t T(t-s)\phi(s)ds, y^*\right\rangle.$$

Putting $y_n^* = n\int_0^{1/n} T^*(s)y^*ds$ (where the integral is taken in the w*-topology), we obtain

$$\limsup_{n\to\infty} \| y_n^*\| = 1, \quad \text{and} \quad y_n^* \to y^*$$

in the w*-topology as $n \to \infty$.

$$\left\langle A\int_0^t T(t-s)\phi(s)ds, y_n^*\right\rangle = \left|\int_0^t \langle \phi(s), T^*(t-s)A^*y_n^*\rangle ds\right|$$

$$\leqq \|\phi\|_{L^\infty}\int_0^t \|T^*(t-s)A^*y_n^*\|ds$$

$$\leqq \beta(t)\|y_n^*\|\,\|\phi\|_{L^\infty}.$$

Taking the limit $n\to\infty$ yields

$$\left\|A\int_0^t T(t-s)\phi(s)ds\right\|\leqq \beta(t)\|\phi\|_{L^\infty}.$$

A standard closedness argument extends this result to all continuous $\phi:[0,\infty)\to Z$. □

COROLLARY 1. *Let Z be the Favard-class of $T(\cdot)$, i.e.*

$$Z=\left\{x\in X\,\middle|\,\limsup_{0\leqq t\leqq 1}\frac{1}{t}\|T(t)x-x\| \text{ is finite}\right\},$$

$$|x|_Z=\|x\|+\limsup_{0\leqq t\leqq 1}\frac{1}{t}\|T(t)x-x\|.$$

Then Z satisfies properties (i) *and* (ii) (see [6]).

COROLLARY 2. *Let Y be a Banach space and let B be a continuous linear operator $Y\to X$. Then there exists a subspace Z of X with a suitable norm $|\cdot|_Z$ such that B is continuous $Y\to Z$ and Z satisfies* (i) (*or, equivalently* (ii)) *iff*

(iv) *there is a continuous, nondecreasing function $\gamma:[0,\infty)\to[0,\infty)$ with $\gamma(0)=0$ and*

$$\operatorname*{Var}_{Y^*}\{B^*T^*(s)x^*;0\leqq s\leqq t\}\leqq \gamma(t)\|x^*\| \qquad \textit{for all } x^*\in X^*$$

or, equivalently, for all $y^*\in D(A^*)$ and $t\geqq 0$

$$\text{(v)}\qquad \int_0^t \|B^*T^*(s)A^*y^*\|_{Y^*}ds\leqq \gamma(t)\|y^*\|.$$

PROOF. Putting $\tilde\gamma(t)=\gamma(t)|B|_{Y,Z}$ shows the implication (ii)→(iv). The converse follows by putting $Z=\text{Range } B$ with norm

$$|By|_Z=\inf_{\substack{y_1\in Y\\ By_1=By}}|y_1|.$$ □

In the remaining part of this section we outline the relations of Theorem 2.1 to other generation results. In [6] the wellposedness of the abstract Cauchy problem

$$\frac{d}{dt}x(t) = (A + K)x(t) \tag{2.1}$$

is discussed where K is an A-bounded, linear operator. As we can write K as $K = BA + F$ with continuous linear operators B and F, (2.1) is well-posed iff $(I + B)A$ generates a C_0-semigroup. The connection between the problems for $A(I + B)$ and $(I + B)A$ is clarified by

THEOREM 2.3. *Let A be a closed, densely defined, linear operator in X and let C be a continuous linear operator on X.*

(i) *If CA generates a C_0-semigroup, so also does AC.*

(ii) *If, in addition, $X_{(AC)^*} = X_{A^*}$, then we have also the converse implication that $\overline{CA}$ generates a C_0-semigroup provided AC does.*

PROOF. Suppose that CA generates a C_0-semigroup $T(\cdot)$ on X. We put for all $x \in X$ and $t > 0$

$$U(t)x = x + A\int_0^t T(s)Cxds.$$

Clearly, $U(0)x = x$ for all $x \in X$. For all positive s, t we obtain

$$\begin{aligned} U(t)U(s)x &= x + A\int_0^s T(\tau)Cxd\tau \\ &\quad + A\int_0^t T(\tau)Cxd\tau + A\int_0^t T(\tau)Cd\tau A\int_0^s T(\sigma)Cxd\sigma \\ &= x + A\int_0^s T(\tau)Cxd\tau + A\int_0^t T(\tau)Cxd\tau \\ &\quad + A\int_0^t T(\tau + s)Cxd\tau - A\int_0^t T(\tau)Cxd\tau \\ &= U(t + s)x. \end{aligned}$$

As $X_{CA} = X_A$ and the mapping $t \to \int_0^t T(s)dsCx$ is continuous $[0,\infty) \to X_A$ we thus conclude that

$$t \to A\int_0^t T(s)Cxds \quad \text{is continuous } [0,\infty) \to X.$$

Consequently, $U(\cdot)$ is a C_0-semigroup on X. So we are able to calculate its infinitesimal generator B.

Choose x so that $Cx \in X_A$. Then

$$\frac{1}{t}(U(t)x - x) = \frac{1}{t} A \int_0^t T(s)Cxds.$$

$(1/t)\int_0^t T(s)Cxds$ converges to Cx as $t \to 0^+$ in the X_A-norm and therefore

$$\frac{1}{t} A \int_0^t T(s)Csds \to ACx \qquad \text{as } t \to 0^+.$$

Thus B is an extension of AC. Conversely, suppose that $(1/t)A\int_0^t T(s)Cxds$ converges to some y as $t \to 0^+$. As $(1/t)\int_0^t T(s)Cxds$ approaches Cx as $t \to 0^+$ and A is closed we obtain $y = ACx$ and so (i) is proved.

In order to verify (ii), suppose that AC generates the C_0-semigroup $U(\cdot)$. For $x \in X_A$, we put for $t \geqq 0$

$$T(t)x = x + C\int_0^t U(\tau)Axd\tau.$$

Clearly, $T(0)x = x$ and the mapping $t \to T(t)x$ is continuous. As $\int_0^t U(\tau)Axd\tau \in X_{AC}$ we deduce that $T(t)x \in X_A$. Moreover, for all s, $t > 0$ we have

$$\begin{aligned} T(t)T(s)x &= x + C\int_0^s U(\tau)Axd\tau \\ &\quad + C\int_0^t U(\tau)d\tau AC\int_0^s U(\sigma)Axd\sigma + C\int_0^t U(\tau)Axd\tau \\ &= x + C\int_0^s U(\tau)Axd\tau + C\int_0^t U(\tau)U(s)Axd\tau \\ &= T(t+s)x. \end{aligned}$$

Finally, we observe that

$$\lim_{t\to 0^+} \frac{1}{t} C\int_0^t U(\tau)Axd\tau = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) = CAx.$$

So it remains to show that the operators $T(t)$ can be extended to a family of continuous linear operators on X that are uniformly bounded on compact t-intervals. This family will constitute a C_0-semigroup on X and by the core-theorem its generator equals $\overline{CA}$. (It is only here where we need that A is densely defined!) For this purpose choose $z^* \in X^*$. Then we have for all $x \in X_A$

and $t \geqq 0$

$$\langle T(t)x, z^*\rangle = \langle x, z^*\rangle + \left\langle Ax, \int_0^t U^*(\tau)C^*z^*d\tau\right\rangle$$

where the integral is taken with respect to the w*-topology. As

$$\left\langle ACy, \int_0^t U^*(\tau)C^*z^*d\tau\right\rangle = \left\langle \int_0^t U(\tau)d\tau ACy, C^*z^*\right\rangle = \langle U(t)y - y, C^*z^*\rangle$$

we conclude that $\int_0^t U^*(\tau)C^*z^*d\tau \in X_{(AC)^*}$ and

$$\left|\int_0^t U^*(\tau)C^*z^*d\tau\right|_{(AC)^*} \leqq M\|z^*\|, \qquad \text{for } 0 \leqq t \leqq T$$

with a suitable chosen constant M.

By hypothesis $X_{(AC)^*} = X_{A^*}$, and hence we have

$$\left|\int_0^t U^*(\tau)C^*z^*d\tau\right|_{A^*} \leqq M_1\|z^*\|.$$

Thus

$$\langle T(t)x, z^*\rangle = \left\langle x, A^*\int_0^t U^*(\tau)C^*z^*d\tau\right\rangle \leqq M_1\|z^*\|\,\|x\|$$

which, in turn, implies that $\|T(t)x\| \leqq M_1\|x\|$, and hence the proof is complete. □

REMARK. Note that in Theorem 2.3(i) CA and not its closure has to be a generator. (Otherwise, the result is clearly wrong!)

The following result may be regarded as a kind of dual result to Theorem 2.1:

THEOREM 2.4. *Let A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X. Let B be a continuous linear operator satisfying*

$$\operatorname*{Var}_X(BT(s)x \mid 0 \leqq s \leqq t) \leqq \gamma(t)\|x\|$$

or, equivalently,

$$\int_0^t \|BT(s)Ax\|\,ds \leqq \gamma(t)\|x\|$$

where γ is a continuous, nondecreasing function $[0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$.

Then both $(I+B)A$ and $A(I+B)$ are the infinitesimal generators of C_0-semigroups on X.

PROOF. Given a strongly continuous family $U(\cdot)$ of continuous linear operators on X we define for all $x \in D(A)$

$$\mu(U)(t)x := T(t)x + \int_0^t U(t-s)BT(s)Axds.$$

Clearly, $\mu(U)(t)x$ depends linearly on x and continuously on t. Moreover,

$$\begin{aligned} \| \mu(U)(t)x \| &\leqq \| T(t)x \| + \int_0^t \| U(t-s) \| \, \| BT(s)Ax \| ds \\ &\leqq \| T(t)x \| + m \int_0^t \left\| \frac{d}{ds} BU(s)x \right\| ds \\ &\leqq \| T(t)x \| + m \operatorname{Var}(BT(s)x \mid 0 \leqq s \leqq t) \\ &\leqq \| T(t)x \| + m\gamma(t) \| x \|, \end{aligned}$$

implying that $\mu(U)(t)$ can be extended to a continuous linear operator on X.

Thus $\mu(U)$ is a strongly continuous family of linear continuous operators on X.

Since

$$\begin{aligned} \| \mu(U)(t)x - \mu(\tilde{U})(t)x \| &\leqq \int_0^t \| U(t-s) - \tilde{U}(t-s) \| \, \| BT(s)Ax \| ds \\ &\leqq \gamma(t) \| x \| \int_0^t \| U(t-s) - \tilde{U}(t-s) \| ds \end{aligned}$$

we can apply a contraction-argument to get a unique fixed point U of μ.

Taking the Laplace transform $\hat{U}(\cdot)$ of U we obtain for all $x \in D(A)$

$$\hat{U}(\lambda)x = (\lambda - A)^{-1}x + \hat{U}(\lambda)B(\lambda - A)^{-1}Ax,$$

i.e.

$$\hat{U}(\lambda)(I - B(\lambda - A)^{-1}A)x = (\lambda - A)^{-1}x,$$

$$\hat{U}(\lambda)(\lambda - A - BA)(\lambda - A)^{-1}x = (\lambda - A)^{-1}x$$

which implies that $\hat{U}(\lambda)(\lambda - A - BA)y = y$ for all $y \in D(A)$. Hence $\hat{U}(\lambda) = (\lambda - A - BA)^{-1}$ and U is the C_0-semigroup with infinitesimal generator $(I+B)A$. By Theorem 2.1 also $A(I+B)$ is a generator. □

3. Feedback for well-posed boundary control problem

The objective of this section is to derive a general semigroup approach to boundary control problems. Again, let Ω be a bounded domain in $\mathbf{R}^n$ with boundary Γ and let X be a Banach space of functions mapping Ω into $\mathbf{R}^n$. Y will denote a Banach space of functions mapping Γ into $\mathbf{R}^n$. Moreover, let $\mathcal{D}$ be a dense linear subspace of X (for instance, $\mathcal{D}(\Omega)$ or $C^\infty(\Omega)$).

Let A be a linear operator in X such that $\mathcal{D} \subset D(A)$ and B be a linear operator $\mathcal{D} \to Y$.

We consider the initial-boundary-value problem

$$\begin{aligned} &\frac{d}{dt}x(t) = Ax(t), \qquad t > 0, \\ &x(0) = x, \\ &Bx(t) = v(t), \qquad t \geqq 0, \end{aligned} \tag{3.1}$$

where the control v belongs to $L^p_{\mathrm{loc}}(0,\infty; Y)$.

Roughly speaking, our goal is to show that if the open loop problem (3.1) is well-posed, then

(i) the homogeneous equation gives rise to a C_0-semigroup,

(ii) there exists a continuous extension operator D such that $x = Dy$ is a (generalized) solution to

$$\begin{aligned} (\lambda - A)x &= 0, \\ Bx &= y, \end{aligned}$$

(iii) the closed loop problem (i.e. (3.1) with the control $v(t) = Fx(t)$) is well-posed.

DEFINITION. A continuous function $\phi : [0,\infty) \to X$ is called a strong solution of (3.1) if there exists a sequence (ϕ_n) of (at least) continuously differentiable functions $[0,\infty) \to X$ such that

$$\begin{aligned} &\phi_n(\cdot) \in \mathcal{D} \qquad \text{for all } t \geqq 0, \\ &\phi_n \to \phi \qquad \text{in } C(0,\infty; X), \\ &\frac{d}{dt}\phi_n(t) - A\phi_n(t) \to 0 \qquad \text{in } L^1_{\mathrm{loc}}(0,\infty; X), \\ &\phi_n(0) \to x \qquad (\text{in } X) \\ &B\phi_n \to v \qquad \text{in } L^p_{\mathrm{loc}}(0,\infty; X). \end{aligned}$$

Here convergence in C means uniform convergence on bounded t-intervals whereas convergence in L^p_{loc} means L^p-convergence on bounded t-intervals.

DEFINITION. The initial-boundary-value problem (3.1) is said to be well-posed for controls in L^p_{loc} if there exists a unique strong solution $\phi(\cdot, x, v)$ of (3.1) for all $x \in X$ and $v \in L^p_{\mathrm{loc}}(0,\infty; Y)$.

We start our considerations with two technical lemmas:

LEMMA 3.1. *Suppose that* (3.1) *is well posed for L^p_{loc}-controls. Then the operator $\mathcal{U}$ given by*

$$\mathcal{U}(v, x) = \phi(\cdot, x, v) \tag{3.2}$$

is a continuous linear operator $L^p_{\mathrm{loc}}(0,\infty; Y) \times X \to C(0,\infty; X)$.

PROOF. By the closed graph theorem ([7], p. 57) it is sufficient to verify that $\mathcal{U}$ is a closed linear operator. This elementary calculation is left to the reader. □

For our further investigations we need exponentially weighted function spaces. Let Z be a Banach space and θ be a real number. We put

$$L^p_\theta(0,\infty; Z) := \left\{ \phi \in L^p_{\mathrm{loc}}(0,\infty; Z) \,\middle|\, |\phi|^p_{p,\theta} = \int_0^\infty e^{-p\theta t} \|\phi(t)\|^p dt < \infty \right\}$$

and

$$C_\theta(0,\infty; Z) := \{\phi \in C(0,\infty; Z) \mid e^{-\theta t}\phi(t) \text{ is bounded and uniformly countinuous on } [0,\infty)\}.$$

The corresponding norm is

$$|\phi|_{\theta,\infty} = \sup_{t \geqq 0} e^{-\theta t} |\phi(t)|.$$

LEMMA 3.2. *Let* (3.2) *be well-posed for L^p_{loc}-controls. Then for each real θ there exists a $\delta > 0$ such that the operator $\mathcal{U}$ given by* (3.2) *maps $L^p_\theta(0,\infty, Y) \times X$ continuously into $C_\delta(0,\infty; X)$.*

(This is just to say that exponentially bounded controls imply that the corresponding solutions are also exponentially bounded.)

PROOF. To begin with, we consider the restrictions of ϕ and v to the interval $J = [0,1]$. Clearly, $\phi\,|_J$ depends only on x and $v\,|_J$. Thus by Lemma 3.1 we infer that the operator $\mathcal{U}_J$ that maps $(v\,|_J, x)$ into $\phi(\cdot, x, v)|_J$ is continuous from $L^p(J; Y) \times X$ into $C(J; X)$. Consequently, there exists a constant $M > 0$ so that for $0 \leqq t \leqq 1$ we have

$$\|\phi(t,x,v)\| \leqq M(\|x\| + \left(\int_0^1 \|v(s)\|^p ds)^{1/p}\right) \qquad \text{for } 0 \leqq t \leqq 1.$$

For $v \in L^p_{\text{loc}}(0,\infty; Y)$ and any positive real m we define $v_m(\cdot)$ by

$$v_m(t) = v(t+m) \qquad \text{for } t \geqq 0.$$

Clearly, $v \in L^p_\theta(0,\infty; Y)$ implies that $|v_m|_{p,\theta} \leqq e^{\theta m} |v|_{p,\theta}$.

The uniqueness of strong solutions implies that for $t \geqq m$

$$\phi(t,x,v) = \phi(t-m, \phi(m,x,v), v_m).$$

Without loss of generality we may assume that $M > 1$, $\theta > 0$ and hence we obtain for $0 \leqq t \leqq 1$

$$\begin{aligned} \|\phi(t,x,v)\| &= \|\phi(t-m, \phi(m,x,v), v_m)\| \\ &\leqq M\left(\|\phi(m,x,v)\| + \left(\int_0^1 \|v_m(s)\|^p ds\right)^{1/p}\right) \\ &\leqq M\left(\|\phi(m,x,v)\| + e^\theta\left(\int_0^\infty e^{-p\theta s}\|v_m(s)\|^p ds\right)^{1/p}\right) \\ &\leqq M(\|\phi(m,x,v)\| + e^\theta e^{m\theta} |v|_{p,\theta}. \end{aligned} \tag{3.3}$$

In particular (3.3) also holds for $t = m+1$.

Choose now $\delta > 0$ such that $Me^{-(\delta-\theta)} < 1$. Then we get

$$e^{-(m+1)\delta}\|\phi(m+1,x,v)\| \leqq Me^{-\delta}e^{-m\delta}\|\phi(m,x,v)\| + Me^{-(\delta-\theta)}e^{-m(\delta-\theta)}|v|_{p,\theta}.$$

An induction argument yields

$$\begin{aligned} e^{-m\delta}\|\phi(m,x,v)\| &\leqq \alpha^m\|x\| + \sum_{j=0}^m \alpha^{m-j}e^{-j(\delta-\theta)}|v|_{p,\theta} \\ &\leqq \alpha^m\|x\| + m\alpha^m|v|_{p,\theta} \end{aligned}$$

with $\alpha = Me^{-(\delta-\theta)}$.

Thus $e^{-m\delta}\phi(m,x,v) \to 0$ as $m \to \infty$, and by (3.3) we infer that $\phi(\cdot,x,v)$ belongs to $C_\delta(0,\infty; X)$.

By Lemma 3.1 $\mathcal{U}$ is a closed linear operator in $L^2_\theta(0,\infty Y) \times X \to C_\delta(0,\infty; X)$ and hence it is continuous. □

DEFINITION. We define an operator $W_0: X \times Y \supseteq D(W_0) \to X$ by $z = W_0(x,y)$ iff $Bx = y$ and $Ax = z$.

LEMMA. 3.3. *Suppose that* (3.1) *is well-posed for* L^p_{loc} *controls. Then the operator* W_0 is closeable in $X \times Y \times X$. *If* W *denotes the closure of* W_0, *then for sufficiently large* $\lambda \in \mathbf{R}$ *and all* $x \in X$, $y \in Y$ *there exists a unique solution* z *to the equation*

$$\lambda z - W(z, y) = x,$$

given by

$$z = \int_0^\infty e^{-\lambda t}\phi(t, x, v)dt,$$

where v *is the constant function* $v(t) = \lambda y$. *Moreover,* z *depends continuously on* x *and* y.

PROOF. We denote by W the closure of W_0 in $X \times Y \times X$, which might be a multivalued operator. (In fact we shall prove finally that it is single valued.) We fix some $\theta > 0$ and choose $\delta > 0$ according to Lemma 3.2. Now let $x \in X$, $y \in Y$, $\lambda > \delta$. We put $v(t) = \lambda y$ and $z = \int_0^\infty e^{-\lambda t}\phi(t, x, v)dt$. Evidently, z depends continuously on x and y. We prove first that z satisfies $\lambda z - W(z, y) = x$.

Let ϕ_n be a sequence in $C^1(0, \infty, X)$ with

$$\phi_n \to \phi(\cdot, x, v) \quad \text{in } C(0, \infty, X), \quad \phi_n' - A\phi_n = g_n \to 0 \quad \text{in } L^1_{\mathrm{loc}}(0, \infty, X),$$

$$B\phi_n = v_n \to v \quad \text{in } L^p_{\mathrm{loc}}(0, \infty, Y).$$

Now we choose a sequence of integers $n_k \to \infty$, such that

$$\int_0^k e^{-\lambda s}\phi_{n_k}(s)ds - \int_0^k e^{-\lambda s}\phi(s, x, v)ds \to 0,$$

$$\int_0^k e^{-\lambda s}v_{n_k}(s)ds - \int_0^k \lambda e^{-\lambda s}y ds \to 0,$$

$$\int_0^k e^{-\lambda s}g_{n_k}(s)ds \to 0,$$

and

$$e^{-\lambda k}\phi_{n_k}(k) - e^{-\lambda k}\phi(k, x, v) \to 0,$$

as k goes to infinity.

Putting

$$z_k = \int_0^k e^{-\lambda s}\phi_{n_k}(s)ds, \qquad y_k = \int_0^k e^{-\lambda s}v_{n_k}(s)ds,$$

we conclude that $z_k \to z$, $y_k \to y$ as $k \to \infty$. Furthermore

$$W_0(e^{-\lambda s}\phi_{n_k}(s), e^{-\lambda s}v_{n_k}(s)) = e^{-\lambda s}(\phi'_{n_k}(s) - g_{n_k}(s)).$$

Using the closedness of the set W we see that

$$W(z_k, y_k) \ni \int_0^k e^{-\lambda s}(\phi'_{n_k}(s) - g_{n_k}(s))ds$$

$$= e^{-\lambda k}\phi_{n_k}(k) - \phi_{n_k}(0) + \lambda z_k - \int_0^k e^{-\lambda s} g_{n_k}(s)ds.$$

Taking the limit for $k \to \infty$ and using again the closedness of W we obtain $W(z, y) \ni \lambda z - x$.

Next we show that the solution z is unique. Of course the proof is sufficient for $x = 0$, $y = 0$. Suppose that $W(z, 0) \ni \lambda z$. Let $z_n \to z$, $y_n \to 0$, $x_n \to 0$ such that $\lambda z_n - W_0(z_n, y_n) = x_n$. Putting $\phi_n(t) = e^{\lambda t} z_n$, we see that

$$\phi_n(t) \to e^{\lambda t} z \qquad \text{in } C(0, \infty, X),$$

$$\phi'_n(t) - A\phi_n(t) = e^{\lambda t} x_n \to 0 \qquad \text{in } L^1_{\text{loc}}(0, \infty, X),$$

$$B\phi_n(t) = e^{\lambda t} y_n \to 0 \qquad \text{in } L^p_{\text{loc}}(0, \infty, Y),$$

$$\phi_n(0) = z_n \to z \qquad \text{in } X.$$

Consequently $\phi(t) = e^{\lambda t} z$ is a strong solution to (3.1) with x replaced by z and $v = 0$. According to Lemma 3.2, the solution has exponential growth $\|\phi(t)\| \leqq Me^{\delta t}$. As $\delta < \lambda$, this implies $z = 0$.

Finally we prove that W is single valued, i.e. that W_0 is closeable. Assume that $W(0, 0) \ni x$. Then for each $\lambda > \delta$, $z = 0$ is the unique solution to $\lambda z - W(z, 0) \ni x$. Therefore $0 = \int_0^\infty e^{-\lambda t}\phi(t, x, 0)dt$. This implies that $\phi(t, x, 0)$ vanishes identically, in particular $x = \phi(0, x, 0) = 0$ □

This lemma has two important consequences. The first one is more or less well known, at least for certain important special cases, but we include it for sake of completeness and self-consistency of the paper.

THEOREM 3.1. *Assume that* (3.1) *is well-posed for L^p_{loc} controls. Then*

$$\mathcal{T}(t)x = \phi(t, x, 0)$$

defines a C_0-semigroup on X. Its infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}x = z \qquad \textit{iff } W(x, 0) = z.$$

PROOF. The continuity of ϕ implies that $\mathcal{T}(t)x$ is continuous in t. By Lemma 3.2 we infer that each $\mathcal{T}(t)$ is a continuous linear operator on X. Evidently $\mathcal{T}(0)x = \phi(0, x, 0) = x$. The semigroup property follows by standard arguments from the uniqueness of the solutions to (3.1). To compute the infinitesimal generator of $\mathcal{T}(t)$, we notice that for sufficiently large λ and all $x \in X$

$$z = (\lambda - \mathcal{A})^{-1}x = \int_0^\infty e^{-\lambda t}\phi(t, x, 0)dt,$$

which is the unique solution to $\lambda z - W(z, 0) = x$. Thus $\mathcal{A} = W(\cdot, 0)$. □

Another consequence of Lemma 3.3 is the well-posedness of the stationary abstract boundary value problem

$$(3.4)\qquad \begin{aligned} (\lambda - A)z &= 0, \\ Bz &= y. \end{aligned}$$

Similarly to the evolution problem we have

DEFINITION. By a strong solution to (3.4) we mean a $z \in X$ such that there exists a sequence z_n in $\mathcal{D}$ with $z_n \to z$ in X, $(\lambda - A)z_n \to 0$ in X and $Bz_n \to y$ in Y.

THEOREM 3.2. *Suppose that* (3.1) *is well-posed for L^p_{loc}-controls. Then for sufficiently large $\lambda \in \mathbf{R}$, there exists a continuous linear operator $D : Y \to X$, such that $z = Dy$ is the unique strong solution to* (3.4).

PROOF. One can easily check that z is a strong solution to (3.4) iff $\lambda z - W(z, y) = 0$. Therefore, Theorem 3.2 is an immediate consequence of Lemma 3.3. □

The next result clarifies the semigroup approach to the boundary control problem.

THEOREM 3.3. *Let* (3.1) *be well-posed for L^p_{loc}-controls. Fix λ sufficiently large and let D be the operator defined above. Then the strong solution ϕ of* (3.1) *can be represented as*

$$\phi(t, x, v) = \mathcal{T}(t)x - \mathcal{A}\int_0^t \mathcal{T}(t-s)Dv(s)ds + \lambda\int_0^t \mathcal{T}(t-s)Dv(s)ds.$$

PROOF. Let $\phi_n \in C^1(\mathbf{R}^+, X)$ be such that $\phi_n \to \phi(\cdot, x, v)$ in $C(\mathbf{R}^+, X)$,

$$g_n = \frac{d}{dt}\phi_n - A\phi_n \to 0 \qquad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^+, X)$$

and

$$v_n = B\phi_n \to v \qquad \text{in } L^p_{\text{loc}}(\mathbf{R}^+, Y).$$

Thus

$$W_0\Big(\phi_n(t), v_n(t)\Big) = -g_n(t) + \frac{d}{dt}\phi_n(t).$$

Using the fact that

$$W(Dv_n(t), v_n(t)) = \lambda v_n(t),$$

we have that

$$\begin{aligned}\mathscr{A}(\phi_n(t) - Dv_n(t)) &= W(\phi_n(t) - Dv_n(t), 0)\\ &= \frac{d}{dt}\phi_n(t) - g_n(t) - \lambda v_n(t).\end{aligned}$$

Consequently

$$\frac{d}{dt}(\phi_n(t) - Dv_n(t)) = \mathscr{A}(\phi_n(t) - Dv_n(t)) - \frac{d}{dt}Dv_n(t) - g_n(t) + \lambda v_n(t);$$

which implies by the variation-of-parameters formula:

$$\begin{aligned}\phi_n(t) - Dv_n(t) &= \mathscr{T}(t)\phi_n(0) - \mathscr{T}(t)Dv_n(0) - \int_0^t \mathscr{T}(t-s)\frac{d}{ds}Dv_n(s)ds\\ &\quad + \int_0^t \mathscr{T}(t-s)g_n(s) + \lambda\int_0^t \mathscr{T}(t-s)v_n(s)ds\\ &= \mathscr{T}(t)\phi_n(0) - \mathscr{T}(t)Dv_n(0) - \frac{d}{dt}\left(\int_0^t \mathscr{T}(s)Dv_n(t-s)ds\right)\\ &\quad + \mathscr{T}(t)Dv_n(0) + \lambda\int_0^t \mathscr{T}(t-s)Dv_n(s)ds + \int_0^t \mathscr{T}(t-s)g_n(s)ds\\ &= \mathscr{T}(t)\phi_n(0) - Dv_n(t) - \mathscr{A}\int_0^t \mathscr{T}(t-s)Dv_n(t-s)ds\\ &\quad + \lambda\int_0^t \mathscr{T}(t-s)Dv_n(s)ds + \int_0^t \mathscr{T}(t-s)g_n(s)ds.\end{aligned}$$

The term $Dv_n(t)$ cancels off.

Taking the limit for $n \to \infty$, we obtain (using the closedness of $\mathscr{A}$):

$$\phi(t, x, 0) = \mathscr{T}(t)x - \mathscr{A}\int_0^t \mathscr{T}(t-s)Dv(s)ds + \lambda\int_0^t \mathscr{T}(t-s)v(s)ds. \qquad \square$$

THEOREM 3.4. *Assume that* (3.1) *is well posed for L^p_{loc}-controls. Choose λ sufficiently large and let D be the operator given as above. Then D satisfies the 'generation condition':*

For each $v \in C(0,\infty; Y)$, $t > 0$,

$$\int_0^t \mathcal{T}(t-s)Dv(s)ds \in D(\mathcal{A}),$$

$$\left\| \mathcal{A}\int_0^t \mathcal{T}(t-s)Dv(s)ds \right\| \leqq \gamma(t) \sup_{s\in[0,t]} \|v(s)\|$$

where $\gamma(t)$ is nondecreasing, continuous, $\gamma(0)=0$.

PROOF. From Theorem 3.3 it is clear that $\int_0^t \mathcal{T}(t-s)Dv(s)ds \in D(\mathcal{A})$. Moreover, we know that there exists some $\theta > 0$ such that

$$e^{-\theta t}\|\phi(t,x,v)\| \leqq M\,|v|_{p,\delta} + M\|x\| \quad \text{for some } \delta>0 \quad \text{(Lemma 3.2)}.$$

Thus for $v \in C(\mathbf{R}^+, Y)$, $t > 0$ we obtain (using the fact that we may put $v(s)=0$ for $s > t$, as $\phi(t,0,v)$ depends only on $v\,|_{[0,t]}$):

$$\begin{aligned}\|\phi(t,0,v)\| &\leqq e^{\theta t}M\left(\int_0^t e^{-p\delta s}ds\right)^{1/p} \sup_{s\in[0,t]}\|v(s)\| \\ &\leqq e^{\theta t}Mt^{1/p}\sup_{s\in[0,t]}\|v(s)\|.\end{aligned}$$

Now

$$\begin{aligned}\left\|\mathcal{A}\int_0^t \mathcal{T}(t-s)Dv(s)ds\right\| &\leqq \left\|\lambda\int_0^t \mathcal{T}(t-s)Dv(s)ds\right\| + \|u(t,0,v)\| \\ &\leqq (\lambda M t e^{\theta t}\|D\| + e^{\theta t}Mt^{1/p})\sup_{s\in[0,t]}\|v(t)\|. \qquad \square\end{aligned}$$

An immediate consequence is

THEOREM 3.5. *Let F be a continuous linear operator $X \to Y$ and suppose that* (3.1) *is well-posed for L^p_{loc}-controls. Then the operator $\mathcal{A}(I+DF)$ is the infinitesimal generator of a C_0-semigroup on X.*

4. Applications

In this section we want to apply the generation results of the previous section to hyperbolic boundary control problems. To begin with, we consider

4.1. *Boundary Feedback for Nonsymmetric Hyperbolic Systems*

Again, let Ω be an open bounded domain in $\mathbf{R}^n$ with boundary Γ. Let $A(x, \partial)$ be a partial differential operator

$$A(x, \partial)u = \sum_{j=1}^{m} A_j(x)\partial_j u + B(x)u.$$

Here $x = (x_1, \dots, x_m) \in \mathbf{R}^m$, $u(x) = (u_1(x), \dots, u_k(x))$ is a k-dimensional vector function of x, ∂_j denotes the partial derivative $\partial/\partial x_j$.

The coefficients A_j and B are smooth $k \times k$-matrix-valued functions defined on Ω.

Given a smooth $l \times k$-matrix-valued function M, and a continuous linear operator $F: L^2(\Omega) \to L^2(\Gamma)$ we consider the mixed boundary control problem

(4.1) $$\frac{\partial u}{\partial t} = A(x, \partial)u \qquad \text{in } \Omega \times [0, T],$$

(4.2) $$u(0) = u_0 \in L^2(\Omega) \qquad \text{in } \Omega,$$

(4.3) $$Mu(t) = Fu(t) \qquad \text{in } \Gamma \times [0, T].$$

The basic goal of our investigations is to show that this mixed boundary feedback problem is well posed in $L^2(\Omega)$.

It is convenient to convert problem (4.1)–(4.3) into a problem on a half-space. This is done by means of local coordinate change and a partition of unity. For details, the reader is referred to [4], [5], for instance.

Having done this, we call the new operator again $\mathscr{A}(x, \partial)$ and denote the new matrices again by M, F, A_j and B, respectively. The region is now

$$\Omega = \{x \in \mathbf{R}^m \mid x = (x_1, x_2, \dots, x_m), x_1 > 0\},$$

with $\partial\Omega = \{x \in \mathbf{R}^m \mid x_1 = 0\}$.

We impose the following standard conditions:

(H1) $A(x, \partial)$ is strictly hyperbolic, i.e. $\Sigma_{j=1}^m A_j \xi_j$ has k distinct real eigenvalues for all nonzero $\xi \in \mathbf{R}^m$ and $x \in \bar{\Omega}$.

(H2) The boundary Γ is non-characteristic, i.e. $\det A_1(x) \neq 0$ for all $x \in \Gamma$.

Let l denote the number of negative eigenvalues of $A_1(x)$ for all $x \in \Gamma_0$.

(H3) The boundary operator M can be written as

$$M = (I, S)$$

where I is the $l \times l$ identity matrix and $S(x)$ is an $l \times k - l$ smooth matrix valued function.

REMARK. Note that $A(x, \partial)$ is in general nonsymmetric and the boundary conditions are *not* assumed to be dissipative.

The main goal of this section is to show that the mixed initial boundary feedback problem is well posed, i.e. (4.1)–(4.3) admits a unique strong solution in $L^2(\Omega)$:

THEOREM 4.1. *For any $u_0 \in L^2(\Omega)$ the mixed problem* (4.1)–(4.3) *has a unique strong solution $u(\cdot)$ belonging to $C(0, T; L^2(\Omega))$.*

PROOF. The main theorem in [17] implies that the associated open loop problem is well-posed for L^2_{loc}-controls and hence the result follows from Theorem 3.5.

On the other hand, the machinery in getting Rauch's result is very sophisticated and therefore it seems to be justified to prove the theorem in a direct and (at least from the semigroup point of view) more transparent way, which will be indicated in the Appendix.

4.2. *Boundary Feedback Problems for Second-Order Hyperbolic Equations*

In this section we are concerned with the feedback acting on the boundary for second order hyperbolic problems.

Let Ω be a bounded open set in $\mathbf{R}^n$ with boundary denoted by Γ. Let $A(x, \partial)$ be a second-order strongly elliptic operator with smooth coefficients.

The problems under consideration are

$$\text{(I)} \qquad \begin{aligned} &\frac{\partial^2 u}{\partial t^2} = A(x, \partial)u \qquad \text{in } \Omega \times (0, T], \\ &u(0) = u_0 \in L^2(\Omega), \\ &\frac{\partial u}{\partial t}(0) = u_1 \in H^{-1}(\Omega), \\ &u(t)\big|_\Gamma = Fu(t), \end{aligned}$$

where F is a continuous linear operator $L^2(\Omega) \to L^2(\Gamma)$, and

$$\text{(II)}\qquad \begin{aligned} &\frac{\partial^2 u}{\partial t^2} = A(x,\partial)u \qquad \text{in } \Omega\times(0,T) \\ &u(0) = u_0 \in L^2(\Omega), \\ &\frac{\partial u}{\partial t}(0) = u_1 \in H^{-1}(\Omega), \\ &u(t)\big|_\Gamma = F\frac{\partial u}{\partial t}(t), \end{aligned}$$

where F is a bounded linear operator $H^{-1}(\Omega)$ into $L^2(\Gamma)$.

REMARK. An important special case of operators F is provided by finite rank feedback, i.e.

$$Fu = \sum_{j=1}^{N} (u, w_j)_{L^2(\Omega)} g_j$$

with $w_j \in L^2(\Omega)$ and $g_j \in L^2(\Omega)$ for problem (I) and $w_j \in D(A^{1/2})$ in case of (II), respectively. Note that specific forms of such finite rank feedbacks are considered in [13] and [18].

The main result of our considerations is

THEOREM 4.2. *The boundary feedback control problems* (I) *and* (II) *are well-posed on* $L^2(\Omega)\times H^{-1}(\Omega)$, *i.e. for any* $u_0 \in L^2(\Omega)$ *and* $u_1 \in H^{-1}(\Omega)$ *there is a unique weak solution* $u(t, u_0, u_1)$.

PROOF. To begin with we collect for the reader's convenience some general properties that are well known in the literature.

Define a linear operator A in $L^2(\Omega)$ by

$$\begin{aligned} &D(A) = \{u \in L^2(\Omega) \,|\, Au \in L^2(\Omega) \quad \text{and} \quad u\big|_\Gamma = 0\}, \\ &Au = A(x,\partial)u. \end{aligned}$$

Then A is the infinitesimal generator of an analytic C_0-semigroup on $L^2(\Omega)$. Moreover, A is also the generator of a sine and cosine family of continuous linear operators on $L^2(\Omega)$ denoted by $S(\cdot)$ and $C(\cdot)$, respectively.

The associated linear operator $\mathcal{A}$,

$$\begin{aligned} &D(\mathcal{A}) = D(A)\times H_0^1(\Omega), \\ &\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \end{aligned}$$

is the infinitesimal generator of a C_0-group $T(\cdot)$ on $H_0^1(\Omega)\times L^2(\Omega)$ which can be represented as

$$T(t)=\begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t)\end{pmatrix}.$$

It is a classical result in semigroup theory that the operator $\mathscr{A}$ with domain $H_0^1(\Omega)\times L^2(\Omega)$ is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)\times H^{-1}(\Omega)$. As we can assume without loss of generality that fractional powers of A are well defined we have $H_0^1(\Omega)=D(A^{1/2})$. We first turn to problem (I). Let D denote the Dirichlet map, the "harmonic" extension of boundary data into the interior given by

$$Dg=y$$

where $A(x,\partial)y=0$ in Ω and $y\,|_\Gamma=g$.

It is a well known result that for all real s, D is a continuous linear operator $H^s(\Gamma)$ into $H^{s+1/2}(\Omega)$.

Making use of this result we can apply semigroup methods to

$$\frac{\partial^2 u}{\partial t^2}=A(x,\partial)u,$$

$$u(0)=u_0,$$

$$\frac{\partial u}{\partial t}(0)=u_1,$$

$$u\,|_\Gamma=g.$$

Putting

$$B=\begin{pmatrix} DF & 0\\ 0 & 0\end{pmatrix}$$

we claim that $\mathscr{A}(I+B)$ generates a C_0-semigroup on $X=L^2(\Omega)\times H^{-1}(\Omega)$. As B is a continuous linear operator in X (in view of the regularity of D) we have to compute B^* in order to use Corollary 2.

Using the representation of $T(\cdot)$ we obtain

$$B^*\mathscr{A}^*T^*(t)=\begin{pmatrix} F^*D^*A^*S^*(t) & F^*D^*C^*(t)A^{*1/2}A^{-1/2}\\ 0 & 0\end{pmatrix}.$$

Note that the pairing is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle_{L^2(\Omega)} + \langle A^{*-1/2} y_1, A^{*-1/2} y_2 \rangle_{L^2(\Omega)}.$$

Thus, for all $x = (x_1, x_2) \in X$ we obtain

$$\int_0^t \| B^* \mathscr{A}^* T^*(s) x \|_X ds$$

$$= \int_0^t \| F^* D^* A^* S^*(s) x_1 \|_{L^2(\Omega)} ds + \int_0^t \| F^* D^* C^*(s) A^{*1/2} A^{-1/2} x_2 \|_{L^2(\Omega)}$$

$$\leqq \text{const} \left(\int_0^t \| D^* A^* S^*(s) x_1 \|_{L^2(\Gamma)} ds + \int_0^t \| D^* C^*(s) A^{*1/2} A^{-1/2} x_2 \|_{L^2(\Gamma)} ds \right)$$

$$= \text{const} \left(\int_0^t [\| D^* A^* S^*(s) x_1 \|_{L^2(\Gamma)} + \| D^* A^{*1/2} C^*(s) A^{-1/2} x_2 \|_{L^2(\Gamma)}] ds. \right.$$

(Note that $D(A^{1/2}) = D(A^{*1/2})$.)

Now, the operators

$$(J_1 x)(t) = D^* A^* S^*(t) x,$$

$$(J_2 x)(t) = D^* A^{*1/2} C^*(t) x$$

are both continuous linear operators $L^2(\Omega)$ into $L^2([0, T] \times \Gamma)$ (see [12], Theorem 2.1). Hence by the Schwartz inequality

$$\int_0^t \| B^* \mathscr{A}^* T^*(s) x \|_X ds \leqq \text{const} \sqrt{t} (\| x_1 \|_{L^2(\Gamma)} + \| A^{*-1/2} x_2 \|_{L^2(\Gamma)})$$

$$= \text{const} \sqrt{t} \| x \|_X$$

which implies the claim.

As the assertion concerning problem (II) is proved along the same lines with B given by

$$B = \begin{pmatrix} 0 & DF \\ 0 & 0 \end{pmatrix}$$

we leave the details to the reader.

Appendix: Sketch of the Proof of Theorem 4.1

As indicated, we will give a direct proof of Theorem 4.1. As the details are lengthy and tedious we shall avoid all the details and restrict ourselves to working out the main ideas.

To begin with, recall that the linear operator A given by

$$D(A) = \{u \in L^2(\Omega) \mid Au \in L^2(\Omega),\ Mu|_\Gamma = 0\},$$
$$Au = A(x, \partial)u \qquad \text{for } u \in D(A)$$

is the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on $L^2(\Omega)$ (see [11]).

Next, define an extension operator D by

$$(4.4) \qquad v = Dg \quad \text{iff } A(x,\partial)v - kv = 0 \text{ in } \Omega \text{ and } Mv = g \text{ in } \Gamma,$$

where k is a positive sufficiently large constant.

If $A(x, \partial)$ is symmetric and M is dissipative the existence of a continuous operator D satisfying (4.4) is established in [8], [19]. In our general case we get

Proposition 4.1. *There exists $k > 0$ such that (4.4) defines a unique linear continuous operator $D : L^2(\Gamma) \to L^2(\Omega)$. In addition there exists a constant C independent of g such that*

$$(4.5) \qquad \| Dg \|_{L^2(\Omega)} + | Dg |_{L^2(\Gamma)} \leqq c \| g \|_{L^2(\Gamma)}.$$

The proof follows closely the arguments given in [10]. As it relies heavily on the theory of pseudodifferential operators we cannot give all the details and refer the reader to [10].

Sketch of the Proof. Suppose that the coefficients of A and M are constants, in fact frozen at their boundary value at $(0, x')$. Applying Fourier transform in x' with the dual variable denoted by $w = (w_2, \dots, w_n)$, we obtain

$$\hat{v} = A_1 \frac{d\hat{v}}{dx_1} + i \sum_{j=2}^{m} A_j w_j \hat{v},$$
$$M\hat{v} = \hat{g}.$$

Consequently, we have

$$\frac{d\hat{v}}{dx_1} = A_1^{-1}\left(\hat{v} - i\sum_{j=2}^{m} A_j w_j \hat{v}\right) =: M(x, w)\hat{v}.$$

If we consider variable coefficients then we get the pseudodifferential version

$$\frac{d\hat{v}}{dx_1} = M(x, w)\hat{v}$$

where $M(x, w) = M(x, D_x)$ is a pseudodifferential operator of the first order.

In the next step we use the Kreiss-symmetrizer, which is constructed locally, i.e. in a conical neighborhood of a boundary point. To be more specific, the symmetrizer is a zero-order pseudodifferential operator with symbol $R(k, x, \omega)$ with L^2-norm independent of k and such that there exist d and $c > 0$ with

(i) R is Hermitian,

(ii) R is homogeneous of degree zero in (k, ω) for $k^2 + \omega^2 \geqq 1$ and is a smooth function in its variables and the coefficients matrix A_j and of S,

(iii) $v^* R v \geqq d\,|v|^2 - c\,|g|^2$ for all vectors v that satisfy the boundary conditions, i.e. $Mv = g$ on Γ,

(iv) $\operatorname{Re} RM \geqq dkI$.

Multiplying both sides of (4.6) by $R(k, x, \omega)$ and taking the inner product with $\hat{v}$, we obtain

$$\left\langle\left(R\frac{d}{dx_1}\hat{v}, \hat{v}\right)\right\rangle = \langle(RM\hat{v}, \hat{v})\rangle$$

where

$$(f, g) := \int_{\mathbf{R}^{n-1}} f(\omega)\overline{g(\omega)}d\omega \quad \text{and} \quad \langle f, g\rangle = \int_0^\infty f(x_1)\overline{g(x_1)}dx_1.$$

Integration by parts with respect to x_1 yields

$$\operatorname{Re}\left\langle\left(R\frac{d}{dx_1}\hat{v}, \hat{v}\right)\right\rangle = \operatorname{Re}\left\langle\left(\frac{d}{dx_1}\hat{v}, R\hat{v}\right)\right\rangle$$

$$= \operatorname{Re}\left((\hat{v}, R\hat{v})\Big|_{x_1=0}^{\infty} - \left\langle\left(\hat{v}, \frac{d}{dx_1}R\hat{v}\right)\right\rangle - \left\langle\left(\hat{v}, R\frac{d}{dx_1}\hat{v}\right)\right\rangle\right).$$

Consequently,

$$2\operatorname{Re}\left\langle\left(R\frac{d}{dx_1}\hat{v}, \hat{v}\right)\right\rangle = -\langle\hat{v}\,|_{x_1=0}, R\hat{v}\,|_{x_1=0}\rangle - \operatorname{Re}\left\langle\left(\hat{v}, \frac{d}{dx_1}R\hat{v}\right)\right\rangle.$$

Condition (iii) implies that

$$2\operatorname{Re}\left\langle\left(R\frac{d}{dx_1}\hat{v}, \hat{v}\right)\right\rangle \leqq -d\,|\hat{v}\,|_{x_1=0}|^2_{L^2(\Gamma)} + c\,|g|^2_{L^2(\Gamma)} + C\,\|\hat{v}\|^2_{L^2(\Omega)}$$

where in the last estimate we used the fact that R is a smooth zero order operator.

On the other hand, we have

$$\operatorname{Re}\langle(RM\hat{v}, \hat{v})\rangle \geqq d\,\|\hat{v}\|^2_{L^2(\Omega)}$$

and combining the last two estimates we arrive at

$$dk \| \hat{v} \|^2_{L^2(\Omega)} \leqq -d \,|\, \hat{v} \,|_{x_1=0} |^2_{L^2(\Gamma)} + c \,|\, g \,|^2_{L^2(\Gamma)} + C \| \hat{v} \|^2_{L^2(\Omega)}$$

for all $\hat{v}$ such that $M\hat{v} = \hat{g}$ on Γ.

Consequently,

$$(dk - C) \| \hat{v} \|^2_{L^2(\Omega)} + d \,|\, v \,|_{x_1=0} |^2_{L^2(\Gamma)} \leqq c \,|\, g \,|^2_{L^2(\Gamma)}.$$

By taking k sufficiently large and using the Parseval equality, we get the energy estimate claimed in the Proposition.

To prove existence of the map D (uniqueness follows immediately from the above estimate), we repeat the same procedure for the adjoint problem and then apply the usual technique of [14].

Our next step is to verify the representation formula for the solution of the boundary–initial value problem:

LEMMA 4.1. *Let u be the solution of the initial-boundary value problem*

$$\frac{d}{dt} u(t) = A(x, \partial) u(t),$$

$$u(0) = u_0,$$

$$Mu = g.$$

Then $u(\cdot)$ can be represented as

$$u(t) = T(t)u_0 - A \int_0^t T(t-s) Dg(s) ds + k \int_0^t T(t-s) Dg(s) ds.$$

The operator $A \int_0^t T(t-s)Dg(s)ds$ is a continuous linear operator $L^2(0, T; \Gamma)$ into $L^2(0, T; L^2(\Omega))$.

The proof follows easily by making use of estimate (1.9) in [10].

So we are to verify that the operator $A(I + DF) + kDF$ is the infinitesimal generator of a C_0-semigroup on X.

Since kDF is a continuous linear operator we can ignore this term when dealing with the generation problem (i.e. we put $k = 0$).

LEMMA 4.2. *For all $x^* \in D(A^*)$ we have*

$$D^* A^* x = A_1^- x^- |_\Gamma$$

where

$$x = (x^-, x^+) \quad \text{and} \quad A_1 = \begin{pmatrix} A_1^- & 0 \\ 0 & A_1^+ \end{pmatrix}.$$

This decomposition is implied by assumption (H2).

PROOF. It is easily verified that the adjoint A^* of A is given by

$$A^*u = -\sum A_j^T(x)\partial_j u - \sum \partial_j A_j^T(x)u \tag{4.5}$$

with $D(A^*) = \{u \in L^2(\Omega) \mid A^*u \in L^2(\Omega)$ and $M^*u\,|_\Gamma = 0\}$ where $M^* = (-(A_1^T)^{-1}S^T A_1^-, I_{k-l})$.

For $y \in D(A^*)$ and $g \in L^2(\Omega)$, we obtain

$$\langle A^*y, Dg\rangle_{L^2(\Omega)} = \langle (A^*(x,\partial)y, Dg\rangle_{L^2(\Omega)}$$

where $A^*(x,\partial)$ is the formal adjoint of $A(x,\partial)$ given by the right hand side of (4.5).

Using Green's formula yields

$$\langle A^*y, Dg\rangle_{L^2(\Omega)} = \langle y, A(x,\partial)Dg\rangle_{L^2(\Omega)} + \langle A_1 y, Dg\rangle_{L^2(\Gamma)}.$$

Now, $A(x,\partial)Dg = 0$ and consequently

$$\langle D^*A^*y, g\rangle_{L^2(\Gamma)} = \langle A_1 y, Dg\rangle_{L^2(\Gamma)}.$$

As y belongs to $D(A^*)$, we conclude that $M^*y\,|_\Gamma = 0$, i.e. y can be written as $y = (y^-, y^+)$ with

$$-(A_1^+)^{-1}S^T A_1^- y^- + y^+ = 0.$$

On the other hand

$$MDg\,|_\Gamma = g$$

is equivalent to

$$Dg^- + S(Dg)^+ = g.$$

Therefore

$$\begin{aligned}\langle A_1 y, Dg\rangle_{L^2(\Gamma)} &= \langle A_1^- y^-, Dg^-\rangle_{L^2(\Gamma)} + \langle A_1^+ y^+, (Dg)^+\rangle_{L^2(\Gamma)}\\ &= \langle A_1^- y^-, g - S(Dg)^+\rangle_{L^2(\Gamma)} + \langle A_1^+(A_1^+)^{-1}S^T A_1^- y^-, (Dg)^+\rangle_{L^2(\Gamma)}\\ &= \langle A_1^- y^-, g\rangle_{L^2(\Gamma)} - \langle S^T A_1^- y^-, Dg^+\rangle_{L^2(\Gamma)} + \langle S^T A_1^- y^-, Dg^+\rangle_{L^2(\Gamma)}\\ &= \langle A_1^- y^-, g\rangle_{L^2(\Gamma)}.\end{aligned}$$

Thus

$$\langle D^*A^*y, g\rangle_{L^2(\Gamma)} = \langle A_1^- y^-, g\rangle_{L^2(\Gamma)}$$

and as this equality holds for $g \in L^2(\Gamma)$ the claim follows.

We next investigate the operator J defined by

$$(Jx)(t) = D^*A^*S^*(t)x.$$

LEMMA 4.3. *J is a continuous linear operator $L^2(\Omega)$ into $L^2(0,T;L^2(\Gamma))$.*

PROOF. We use the regularity result from Chang [4] stating that there exists a constant c such that

$$\int_0^T |\, T^*(t)x\,|_\Gamma\,|^2_{L^2(\Gamma)}dt \leqq c\,|\,x\,|^2.$$

With this estimate in mind, we obtain making use of Lemma 4.2

$$\begin{aligned}\int_0^T |D^*A^*T^*(t)x\,|^2_{L^2(\Gamma)}dt &\leqq \int_0^T |A_1^-(T^*(t)x)^-|^2_{L^2(\Gamma)}dt\\ &\leqq \text{const}\int_0^T |(T^*(t)x)^-|^2_{L^2(\Gamma)}dt\\ &\leqq \text{const}\int_0^T |\,T^*(t)x\,|^2_{L^2(\Gamma)}dt\\ &\leqq \text{const}\,|\,x\,|^2.\end{aligned}$$

PROOF OF THE THEOREM. With $X = L^2(\Omega)$ and $B = DF$ the 'generation condition' becomes

$$\int_0^t |\,B^*A^*T^*(s)x\,|_{L^2(\Omega)}ds \leqq \gamma(t)\|\,x\,\|.$$

The left hand side becomes for our example

$$\begin{aligned}\int_0^t |\,F^*D^*A^*T^*(s)x\,|ds &\leqq \text{const}\int_0^t |\,D^*A^*T^*(s)x\,|\,ds\\ &\leqq \text{const}\left(\int_0^t |\,D^*A^*T^*(s)x\,|^2ds\right)^{1/2}t^{1/2}\\ &\leqq \text{const}\,t^{1/2}\|\,x\,\|\end{aligned}$$

(by Lemma 4.4). □

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References

1. J. Ball, *Strongly continuous semigroups, weak solutions and the variation of constants formula*, Proc. Am. Math. Soc. **63** (1977), 370–373.

2. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, North-Holland, 1976.

3. H. Brezis, *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, North-Holland, 1973.

4. S. Chang, *Riccati equations for nonsymmetric and nondissipative hyperbolic systems*, Ph.D. Thesis, Univ. of Florida, 1984.

5. J. Chazarain and J. Piriou, *Introduction a la theorie des equations aux derivees partielles lineaires*, Gauthier-Villars, 1981.

6. W. Desch and W. Schappacher, *On relatively bounded perturbations of linear C_0-semigroups*, Ann. Sci. Norm. Super. Pisa **11** (1984), 327–341.

7. N. Dunford and J. Schwartz, *Linear Operators I*, Interscience, 1957.

8. K. O. Friedrichs and P. Lax, *Boundary value problems for first order operators*, Comm. Pure Appl. Math. **18** (1965), 355–388.

9. T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1976.

10. H. O. Kreiss, *Initial boundary value problems for hyperbolic equations*, Comm. Pure Appl. Math. **13** (1970), 277–298.

11. I. Lasiecka and R. Triggiani, *A cosine operator approach to modelling $L_2(0, T, L^2(\Gamma))$-boundary input hyperbolic equations*, Appl. Math. Optim. **7** (1981), 35–93.

12. I. Lasiecka and R. Triggiani, *Regularity of hyperbolic equations with $L_2(0, T; L_2(\Gamma))$ Dirichlet boundary terms*, Appl. Math. Optim. **10** (1983), 275–286.

13. I. Lasiecka and R. Triggiani, *An L^2 theory for the quadratic optimal cost problem of hyperbolic equations*, in *Control Theory for Distributed Parameter Systems and Applications* (F. Kappel, K. Kunisch and W. Schappacher, eds.), Springer Lecture Notes in Control and Information Sciences **54** (1983), 138–152.

14. P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math. **13** (1960), 427–455.

15. A. Majda and S. Osher, *Initial boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. **28** (1975), 607–676.

16. A. Pazy, *Linear Semigroups and Applications*, Springer, 1983.

17. J. Rauch, *L^2 is a continuable initial condition for Kreiss' Problems*, Comm. Pure Appl. Math. **25** (1972), 265–285.

18. R. Triggiani and I. Lasiecka, *Boundary feedback stabilization problems for hyperbolic equations*, in *Control Theory for Distributed Parameter Systems and Applications* (F. Kappel, K. Kunisch and W. Schappacher, eds.), Springer Lecture Notes in Control and Information Sciences **54** (1983), 238–245.

19. R. Vinter and T. Johnson, *Optimal control of nonsymmetric hyperbolic systems in n-variables in half spaces*, SIAM J. Control **15** (1977), 129–143.